An effective proof of the p-curvature conjecture for order one linear differential equations

ongoing joint work with Florian Fürnsinn.

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GADEPs: Differential Equations and Prime Numbers

Power series hierarchy

$$y = \sum_{n \geq 0} u_n x^n, \qquad u_n \in \mathbb{Q}$$

Rational

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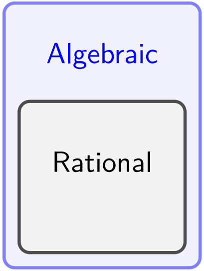
y is algebraic if $\exists P \in \mathbb{Z}[x, T],\ P(x, y) = 0.$



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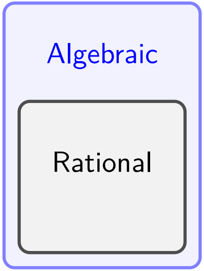

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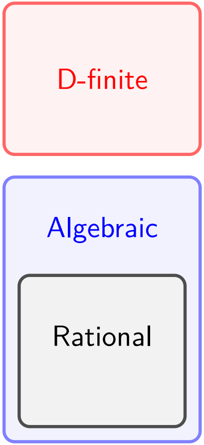

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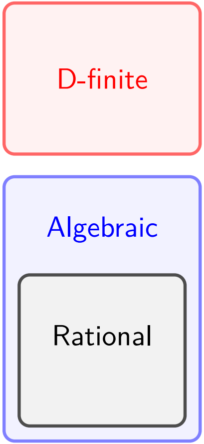

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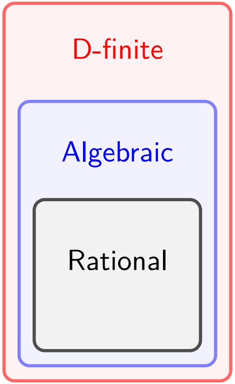

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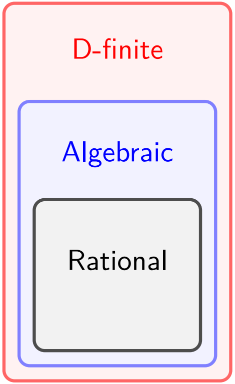

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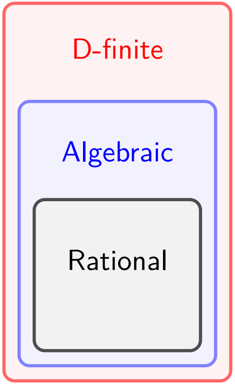
What D-finite power series are algebraic?



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Let $\mathcal{L} = a_n \left(\frac{d}{dx}\right)^n + \cdots + a_1 \frac{d}{dx} + a_0$, $a_i \in \mathbb{Z}[x]$.
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[Singer, 1980], relying on Risch's algorithm.

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All solutions of $\mathcal{L}y = 0$ are algebraic over $\mathbb{Q}(x)$ *if and only if* for almost all prime numbers p, $\mathcal{L}_p y = 0$ has a basis of algebraic solutions over $\mathbb{F}_p(x)$.

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$\to$ Observation: when the solution is not algebraic, p-curvatures are often nonzero.
For $(x^2+1)y' + y = 0$, p-curvatures are $\frac{1}{x^4+1} \in \mathbb{F}_2(x)$, $-\frac{1}{x^6+1} \in \mathbb{F}_3(x)$.

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- Not a decision procedure: infinitely many conditions to check.

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Fact: $y = \exp\left(\int \sum_i \frac{\alpha_i}{x-\beta_i}\right) = \prod_i (x - \beta_i)^{\alpha_i}$ is algebraic *if and only if* $\forall i, \alpha_i \in \mathbb{Q}$.

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Theorem (Jacobson, 1937)

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- Let $u = \sum_i \frac{\alpha_i}{x - \beta_i}$, its p-curvature is $\sum_i \frac{\alpha_i^p - \alpha_i}{(x - \beta_i)^p}$, vanishes *if and only if* [1] $\alpha_i \in \mathbb{F}_p$.

The proof of Honda's Theorem is reduced to proving that for a single $\alpha \in \overline{\mathbb{Q}}$, α is rational *if and only if* α mod $p \in \mathbb{F}_p$ for almost all primes p.

- α mod p is a root of π_α mod p, with $\pi_\alpha \in \mathbb{Z}[x]$ the minimal polynomial of α.

[1] Except for primes p dividing $\beta_i - \beta_j$, $i \neq j$.

Kronecker's theorem

Theorem (Kronecker, 1880; Chebotarev, 1926)

Let $R \in \mathbb{Q}[x]$ be irreducible. If for almost all prime numbers p the polynomial $R \bmod p$ has a root in $\mathbb{F}_p$, then R has a root in $\mathbb{Q}$, hence is linear.

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The p-curvature conjecture holds for equations $y' = \frac{a}{b}y$ with $a, b \in \mathbb{Q}[x]$.

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Proof: Honda's Theorem is equivalent to Kronecker's Theorem.

Theorem (Equivalent statement of Kronecker's Theorem)

Let $R \in \mathbb{Q}[x]$. If for almost all prime numbers p the reduction of R modulo p splits completely over $\mathbb{F}_p$, then R splits completely over $\mathbb{Q}$.

Effective Kronecker

Theorem (Chudnovsky^2, 1985)

Let $R \in \mathbb{Z}[w]$ with leading coefficient $\Delta \in \mathbb{Z}$. There exists $\sigma \in \mathbb{N}$ such that R splits completely over $\mathbb{Q}$ if and only if $R \bmod p$ splits completely over $\mathbb{F}_p$ for all primes p:

- *not dividing Δ,*
- *at most σ.*

Effective Kronecker

Theorem (Chudnovsky[2], 1985; Fürnsinn-P., 2025+)

Let $R \in \mathbb{Z}[w]$ with leading coefficient $\Delta \in \mathbb{Z}$, and let $t(\Delta) := \prod_{p|\Delta} p^{1/(p-1)}$.
Let $B \in \mathbb{R}$ be an upper bound on the modulus of all complex roots of R.
Let $M := \lceil 2.826 \cdot \Delta^3 \cdot t(\Delta) \rceil$ and $N := \lceil 10BM \rceil$.
Then R splits completely over $\mathbb{Q}$ if and only if $R \bmod p$ splits completely over $\mathbb{F}_p$ for all primes p:

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Criterion: If $p \le \sigma$, $p \nmid \Delta$ and $R \bmod p$ does not split completely in $\mathbb{F}_p$, then R does not split completely in $\mathbb{Q}$.

Padé approximation

Given power series $f_1, \ldots, f_r \in \mathbb{Q}[[x]]$ and $n, s \in \mathbb{N}$, find polynomials $P_i \in \mathbb{Q}[x]$ such that $\deg(P_i) \leq n$ and

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Algebraicity criterion: With $f_i = f^{i-1}$, f is algebraic *if and only if* the remainder $P_1 + P_2 f + \cdots + P_r f^{r-1}$ vanishes for large n, r.

Chudnovskys' proof

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Effectivity

Lemma

Let $k, \ell, r \in \mathbb{N}$, $p \not| \operatorname{den}(\alpha)$ a rational prime, $\mathfrak{p}$ a prime ideal of $\mathbb{Q}(\alpha)$ above p.

i. If $\alpha \bmod \mathfrak{p} \in \mathbb{F}_p$, then p does not divide the denominator of $\binom{k\alpha+\ell}{r}$.

ii. If $\alpha \bmod \mathfrak{p} \notin \mathbb{F}_p$ and p divides the denominator of $\binom{k\alpha+\ell}{r}$, then $p \leq r$.

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Theorem (Rothstein-Trager, 1976)

Let $a, b \in \mathbb{Q}[x]$ with b squarefree. The roots of the polynomial $\mathrm{res}_x(b, a - wb') \in \mathbb{Q}[w]$ are exactly the residues of the rational function a/b.

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- $R := \mathrm{res}_x(b, a - wb') \in \mathbb{Q}[w]$ is called Rothstein-Trager's resultant.

Effective Honda

Corollary [Chudnovsky[2], 1985; Fürnsinn-P., 2025+]

Let $a, b \in \mathbb{Z}[x]$, $\deg(a) < d := \deg(b)$ and $R := \mathrm{res}_x(b, a - wb') \in \mathbb{Q}[w]$, with leading coefficient $\Delta := \mathrm{res}_x(b, -b')$, $t := \prod_{p|\Delta} p^{1/(p-1)}$.
Let $B \in \mathbb{R}$ be an upper bound on the modulus of all complex roots of R.
Let $M := \lceil 2.826 \cdot \Delta^3 \cdot t(\Delta) \rceil$ and $N := \lceil 10BM \rceil$.
All solutions of $y' = \frac{a}{b} y$ are algebraic *if and only if* the p-curvatures of the differential equation vanish for all primes p:

- not dividing Δ;
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First algorithm and complexity

Input $a, b \in \mathbb{Z}[x]$ with b squarefree and $\deg(a) < \deg(b)$.

Output The nature (algebraic or transcendental) of $\exp(\int \frac{a}{b})$.

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1. $R := \mathrm{res}_x(b, a - wb') \in \mathbb{Q}[w]$, Δ, t, B;
2. $M := \lceil 2.826 \cdot \Delta^3 \cdot t(\Delta) \rceil$, $N := 10BM$, $\sigma := (2M+1)N + 2M$, $p \leftarrow 2$;
3. **while** $p \leq \sigma$:
   i. **if** $p \nmid \Delta$, **then** compute the p-curvature;
   ii. **if** p-curvature $\neq 0$, **then** return transcendental, **else** $p \leftarrow \mathrm{nextprime}(p)$;
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Theorem (van Hoeij-Novocin, 2012)

Factorization of a monic univariate polynomial $R \in \mathbb{Z}[w]$ of degree d, whose coefficients are bounded by $A \in \mathbb{N}$ can be done in

$$\tilde{O}(d^6 + d^5 \log(A))$$

bit operations where the notation $\tilde{O}$ hides logarithmic factors $\log(d)$ *and* $\log(\log A)$.

Algorithmic

- How to compute p-curvatures?

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- How to compute p-curvatures? [Bostan-Schost, 2009], [Pagès, 2021]

a	b	σ	Output	
$2x+1$	x^2+x+1	3126	algebraic	$a = b'$
$3x-4$	$2x^2-6x+4$	344965	algebraic	
x^2+2x-1	x^3+x+1	276012	transcendental	5-curvature $\neq 0$

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Theorem (Vaaler-Voloch, 2000)

Let $L/\mathbb{Q}$ be a Galois extension of degree ℓ and discriminant Δ_L. Assume GRH. If $\Delta_L \geq \frac{1}{8}\exp(2(\ell-1)\max(105, 25\log\ell))$, then there exists a prime $p \in \mathbb{Z}$ such that:

- $p \nmid \Delta_L$;
- *p does not split completely in L;*
- $p \leq 26\ell^2\Delta_L^{\frac{1}{2(\ell-1)}}$.

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Thank you.